

VSRT INTRODUCTION

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PREFACE

The Very Small Radio Telescope (VSRT) grew out of an NSF supported project at Haystack Observatory in Westford, MA. The project was to develop inexpensive and portable radio astronomy related tools and materials that assist in the teaching of science, engineering, mathematics, and astronomy in support of the national STEM education goals. The original target audience was community college faculty and students at Middlesex Community College (Massachusetts) and Roane State Community College (Tennessee) but the materials are easily adapted for use in high school and other undergraduate institutions as well.

The basic system is a 2-element interferometer with manual pointing that can be used outside to make observations of the Sun or inside as a laboratory apparatus for demonstrating the principles of radio interferometry and basic physics. Students can use the apparatus and accompanying data analysis to study topics like polarization, reflection, and refraction of microwaves and the opacity of various materials to radio emission.

This *VSRT Instruction Manual* is intended to provide an introduction to the theory behind the VSRT. Detailed instructions on assembling VSRT hardware can be found in the *VSRT Assembly Manual*. Directions for installing and running the necessary VSRT software are online at haystack.mit.edu (VSRT_Team), as are several lesson plans using the VSRT.

INTRODUCTION TO INTERFEROMETRY

Most radio telescopes you have probably seen involve one curved reflective surface that collects light from a distant source. The larger the reflective surface, the more light you can collect, and the fainter the objects you can see. As the wavelength you are interested in gets larger and larger



Figure 1: Arecibo Observatory (NAIC - Arecibo Observatory)

(remember, radio wavelengths range from centimeters to kilometers), you need to make your reflective surface larger and larger if you want to see any details of the object you are observing. Unfortunately, technology limits how large we can make these reflective surfaces – think of the weight alone of a really large radio telescope. One of the largest single dish radio telescopes, Arecibo in Puerto Rico – with a diameter of about 300 meters – is built into the ground and can only observe sources to within 20 degrees of straight up. (Figure 1: Arecibo Observatory)

One way to increase the light gathering power of a radio telescope is to connect more than one reflecting surface together to act as an interferometer. The Very Large Array (VLA) in New Mexico (Figure 2) is an interferometer made up of 27 telescopes, each with a diameter of 25 m. When used together, the VLA dishes can see detail as though it were a single dish with a diameter of 36 km. (VLA, Wikipedia)



Figure 2: The Very Large Array (NRAO/AUI - VLA)

Radio astronomers are used to combining signals from more than one antenna to obtain data from radio sources. The VSRT is an example of an instrument that combines signals by adding them; therefore, it is useful to understand how we combine signals from multiple telescopes observing the same object at the same time.

BASIC SIGNAL THEORY

Since light is a wave, and radio signals are a form of light, it is useful to review – both mathematically and graphically - how waves add.

Let's say we have wave $S_1(t)$

$$S_1(t) = \sin(\omega t)$$

And a wave $S_2(t)$ that arrives a time Δt later

$$S_2(t) = \sin(\omega t - \omega \Delta t)$$

where $\omega = 2\pi f$, f is frequency in Hz, ω is frequency in radians/second, and t is in seconds. Note that if $S_2(t)$ was arriving earlier, you would add $\omega\Delta t$.

Combining $S_1(t)$ and $S_2(t)$ together into $s(t)$ we find

$$s(t) = S_1(t) + S_2(t) = \sin(\omega t) + \sin(\omega t - \omega\Delta t)$$

There are times when these waves will constructively interfere or destructively interfere. Let us look at these different options graphically. In the graphs that follow, $S_1(t)$ is in **blue**, $S_2(t)$ is in **red**, and the sum $s(t)$ is in **purple**.

Constructive interference occurs when the two waves are in phase – i.e. when the maxima (or, similarly, the minima) of the two waves overlap. Below is an example of constructive interference, where $S_1(t)$ and $S_2(t)$ are in phase. Note that as a result of being in phase, the **red** and **blue** waves are graphically on top of each other, making them hard to see as individual waves. Notice that in this case, the sum $s(t)$ (in **purple**) is higher in amplitude than both $S_1(t)$ and $S_2(t)$ and has a maximum value of 2.

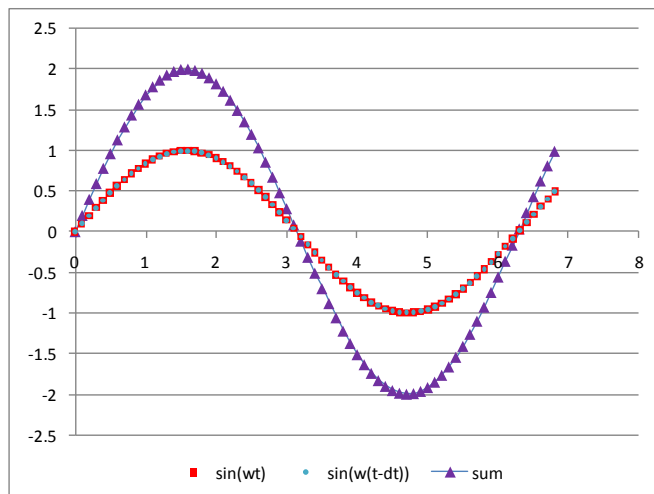


Figure 3: Constructive interference

Destructive interference occurs when the two waves are not in phase, and they effectively cancel each other out – one wave’s maximum overlaps the other’s minimum. Below is an example of destructive interference, where $S_1(t)$ and $S_2(t)$ are 180° out of phase. The net result is that $s(t) = 0$.

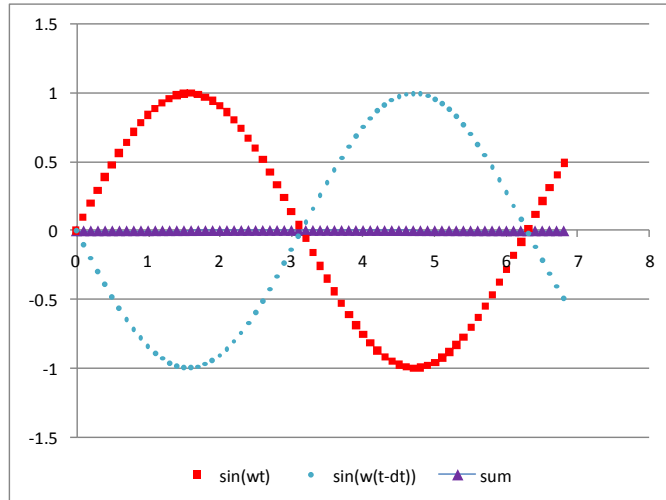


Figure 4: Destructive Interference

Below is an example of two waves that are out of phase by 45° . Notice that at any given time, the **red** wave lags the **blue** due to the phase difference, and the maximum for $s(t)$ (in **purple**) is always less than 2, the max value in the example of constructive interference described above.

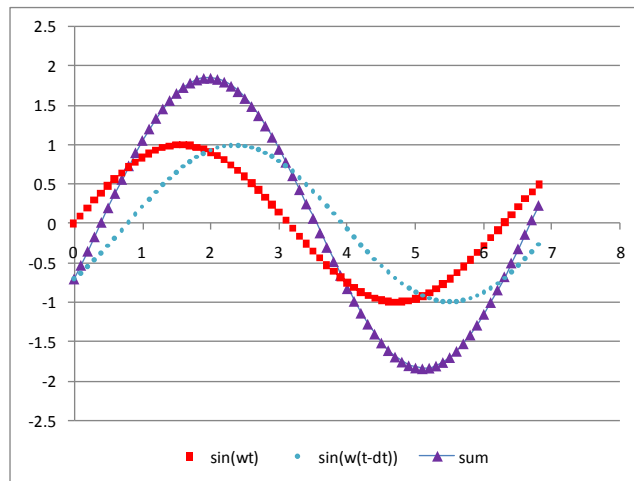


Figure 5: Integrating two waves 45 degrees out of phase

If we apply the explanation above to interferometers, we can learn a lot about how the VSRT works. In what we will call a “simple interferometer”, two collectors gather waves of the same frequency and add them together. The simple interferometer can be arranged to obtain destructive or constructive interference.

The VSRT is not a simple interferometer – it has two collectors, but the frequencies collected by each are changed slightly before adding them together. This frequency shift has a practical purpose: microwave frequencies we are detecting are on the order of 10^{10} Hz and the equipment needed to measure these frequencies is prohibitively expensive. By shifting the frequencies before adding them with the VSRT, we instead measure the resulting beat frequency, which is on the order of 10^6 Hz, which can be done with our less expensive equipment. The frequency difference depends on the particular units and may need to be adjusted if the beat frequency is not between and about 15 and 1000 kHz.

A byproduct of this frequency shift is that when you use the VSRT with a single CFL (compact fluorescent light bulb) source – which is a source of microwave radiation - you cannot get destructive or constructive interference, though with two CFL sources you can. Uncovering why this is the case is an excellent way to explore how the VSRT works, but does involve quite a bit of math, outlined below. You can, however, safely skip ahead to the assembly of the instrument and come back to this section later if you want to learn more about how the VSRT integrates signals.

Useful trigonometric identities for the following calculations are summarized in Table 1.

Table 1: Helpful Trigonometric Identities

Trig Identity 1	$\sin(A) + \sin(B) = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$
Trig Identity 2	$\sin\left(\frac{C}{2}\right) = \pm\sqrt{\frac{1-\cos(C)}{2}}$ so that $\sin^2\left(\frac{C}{2}\right) = \frac{1-\cos(C)}{2}$
Trig Identity 3	$\cos\left(\frac{D}{2}\right) = \pm\sqrt{\frac{1+\cos(D)}{2}}$ so that $\cos^2\left(\frac{D}{2}\right) = \frac{1+\cos(D)}{2}$
Trig Identity 4	$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

CASE 1: A SIMPLE INTERFEROMETER

We define a simple interferometer as two collectors, distance D apart, looking at a source that is an angle θ from the normal. We assume that light is traveling at c and is coming from a single source that is far enough away that the wave fronts are parallel, as show in Figure 6. We also assume that the signals have the same frequency.

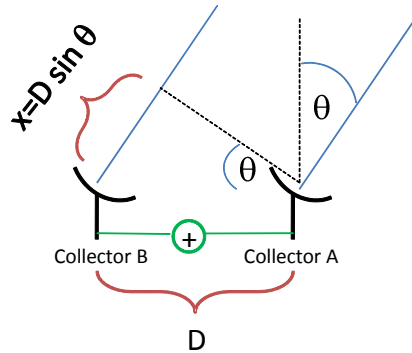


Figure 6: Simple interferometer

Because the source is not necessarily directly overhead, the wave received by the collector on the left has to travel an additional distance x , and is delayed by Δt . Using the figure above, the basic definitions of $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$, $c = \lambda f$, and $c\Delta t = x$ we can express this additional distance x as

$$x = D\sin\theta = c\Delta t = \frac{\lambda\omega}{2\pi}\Delta t.$$

Select values for x are ideal for constructive interference, and others are ideal for destructive interference. Note that as long as D , θ , and c are constant, so is Δt .

Conditions for Destructive Interference

Applying trig identity 1 from table 1 to our original expression for $s(t)$, we find that

$$s(t) = 2\sin\left[\frac{2\omega t - \omega\Delta t}{2}\right]\cos\left[\frac{\omega\Delta t}{2}\right].$$

Putting $s(t)$ in this form allows us to see that the sine term depends on t and the cosine term does not – it depends on Δt , a constant, instead. As a result, there are values of Δt for which the cosine term – and therefore $s(t)$ - will always be zero, the condition for destructive interference.

The $\cos\left[\frac{\omega\Delta t}{2}\right]$ term will be zero when the argument is some odd multiple of $\pi/2$ (recall that $\pi/2$ in radians is equivalent to 90°) or when:

$$\frac{\omega\Delta t}{2} = \frac{(2n + 1)\pi}{2}$$

where $n=0,1,\dots$

Solving for Δt we find

$$\Delta t = \frac{(2n + 1)\pi}{\omega}.$$

If we plug this expression for Δt into our expression for x , we find

$$x = \frac{\lambda\omega}{2\pi} \Delta t = \frac{\lambda\omega}{2\pi} \frac{(2n + 1)\pi}{\omega} = \frac{(2n + 1)}{2} \lambda = \left(n + \frac{1}{2}\right) \lambda$$

In other words, we will have destructive interference when the two sine waves arrive an odd number of half wavelengths apart, equivalent to being of phase by 180° .

Conditions for Constructive Interference

In our expression for $s(t)$

$$s(t) = 2\sin\left[\frac{2\omega t - \omega\Delta t}{2}\right] \cos\left[\frac{\omega\Delta t}{2}\right]$$

we will have constructive interference when the $\left|\cos\left[\frac{\omega\Delta t}{2}\right]\right|$ term is a maximum. This occurs when

$$\frac{\omega\Delta t}{2} = n\pi$$

or when

$$x = D\sin\theta = n\lambda.$$

Therefore, when the distance x is an even multiple of half wavelengths (or an integer number of wavelengths, n), we will have constructive interference.

Interestingly, since $x = D\sin\theta$, the distance D between the collectors AND the location θ of the source are important factors that affect what kind of interference occurs. Therefore, with a simple interferometer observing at a given frequency, we could in principle simply separate the collectors (affecting D) or change the source direction (θ) to produce constructive or destructive interference.

CASE 2: VSRT WITH A SINGLE LIGHT SOURCE

The VSRT is NOT a simple interferometer because it adds waves with slightly different frequencies. The VSRT has two collectors, or *Low Noise Block Downconverter Feeds* (LNBFs). Each LNBF (A and B), downconverts the collected signal frequency by ω_a and ω_b respectively. For example, an LNBF signal at 12 GHz is mixed with a local oscillator at 11.25 GHz so that it is downconverted to the difference ($\omega_b - \omega_a$) of the two frequencies, 750 MHz. As a result of these shifts, the signals that exit the LNBFs take the following form:

$$\begin{aligned} S_A(t) &= \sin[(\omega - \omega_a)t] \\ S_B(t) &= \sin[(\omega - \omega_b)t - \omega\Delta t] \end{aligned}$$

If we combine S_A and S_B , we have

$$s_{VSRT}(t) = S_A(t) + S_B(t) = \sin[(\omega - \omega_a)t] + \sin[(\omega - \omega_b)t - \omega\Delta t]$$

Applying trig identity 1 listed in table 1, we find that

$$s_{VSRT}(t) = 2\sin\left[\frac{2\omega t - \omega\Delta t - (\omega_a + \omega_b)t}{2}\right] \cos\left[\frac{\omega\Delta t + (\omega_b - \omega_a)t}{2}\right].$$

The terms highlighted in yellow are where $s_{VSRT}(t)$ differs from $s(t)$ from the simple interferometer. The sine term still depends on t , but the cosine term now also depends on t . Therefore, unless $\omega_a = \omega_b$, the cosine term varies with time. Note that if $\omega_a = \omega_b = 0$ then $s_{VSRT}(t) = s(t)$.

Therefore, because the LNBFs convert the incoming frequencies down to slightly different output frequencies, the VSRT does not behave exactly as the simple baseline interferometer described in CASE 1. So what kind of signal does the VSRT put out?

After the frequency shifted signals are combined into $s_{VSRT}(t)$, the signal is passed through a Square Law Detector, where the signal is squared:

$$s_{VSRT}^2(t) = 4\sin^2\left[\frac{2\omega t - \omega\Delta t - (\omega_a + \omega_b)t}{2}\right] \cos^2\left[\frac{\omega\Delta t + (\omega_b - \omega_a)t}{2}\right]$$

Using trig identities 2 and 3 from table 1, we find:

$$s_{VSRT}^2(t) = \{1 - \cos[2\omega t - \omega\Delta t - (\omega_a + \omega_b)t]\}\{1 + \cos[\omega\Delta t + (\omega_b - \omega_a)t]\}$$

There are several terms in this expression and since this signal is integrated over time, it is useful to investigate which terms contribute most to the overall signal. The order of magnitude of each of the terms is listed in Table 2.

Table 2

Term	Typical Values	Order of Magnitude
ω	$2\pi \cdot 12 \times 10^9 \text{Hz}$	10^{11} rad/s
ω_a and ω_b	$2\pi \cdot 11.25 \times 10^9 \text{Hz}$	10^{11} rad/s
$\omega_b - \omega_a$	$2\pi \cdot 3 \times 10^5 \text{Hz}$	10^7 rad/s
$\omega_a + \omega_b$	$2\pi \cdot 22.5 \times 10^9 \text{Hz}$	10^{11} rad/s
Δt	$x/c = 10 \text{ m}/3 \times 10^8 \text{ m/s}$	10^{-7} s
$\omega \Delta t$	$2\pi \cdot 12 \times 10^9 \text{Hz} \cdot 10^{-7}$	10^4 rad

Recall that f (in Hz) = $\omega/2\pi$. If we plug these values in and only keep the dominant factors, we find

$$s_{VSR}^2(t) = [1 - \cos(\sim 10^{11} t)][1 + \cos(\sim 10^7 t)]$$

or

$$s_{VSR}^2(t) = 1 + \cos(\sim 10^7 t) - \cos(\sim 10^{11} t) - \cos(\sim 10^{11} t) \cos(\sim 10^7 t).$$

As a result of this simplification, the arguments for the cosine terms are either a high (10^{11}) or a low (10^7) angular frequency. If we plot sample cosines with high (in red) and low frequencies (in blue) on the same plot, we get Figure 7:

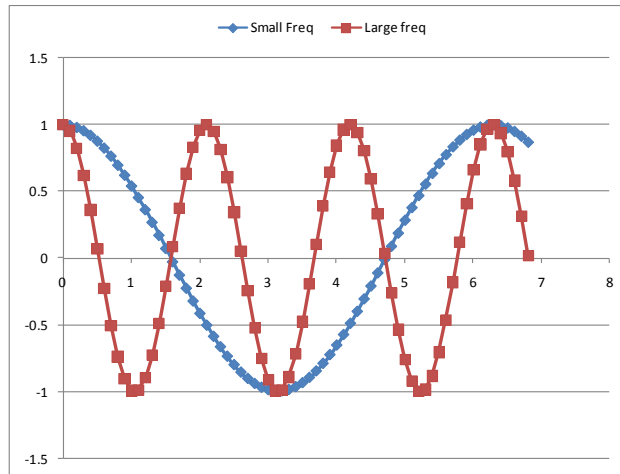


Figure 7: Comparison of waves with large and small frequencies

Notice that over time, the high frequency (red) curve oscillates many more times around zero than does the low frequency (blue) curve over the same period of time. Therefore, the high frequency plot will average out to zero faster than the low frequency.

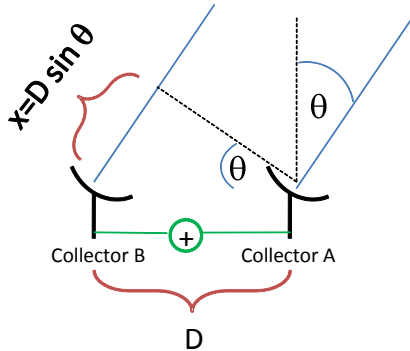
Given that the high frequency terms will average out to zero faster over time, any term containing a high frequency in the expression for $s_{VSRT}^2(t)$ will go to zero and we are left with

$$s_{VSRT}^2(t) = 1 + \cos[\omega\Delta t + (\omega_b - \omega_a)t].$$

We can interpret this expression as a constant plus a sinusoidal term that depends on the beat frequency $(\omega_b - \omega_a)$ between the two oscillators. Again, this beat frequency is on the order of 10^6 Hz, within the range of our existing electronics. Note that since the cosine term depends on time, there is no time when you can get destructive or constructive interference using the VSRT and a single CFL source.

Table 3 summarizes the similarities and differences between a simple interferometer and a VSRT with a single source.

Table 3: Summary of Signal Integration with a single CFL



- single source infinitely far away at an angle θ (source position measured as angular deviation from normal)
- two collectors a distance D apart

Simple Interferometer with a Single Source	VSRT with a Single CFL
<p><i>Signals are simply added</i></p>	<p><i>Signals are frequency shifted then added</i></p>
<p>Summed Wave: $s(t) = S_1(t) + S_2(t)$ $s(t) = 2\sin\left[\frac{2\omega t - \omega\Delta t}{2}\right] \cos\left[\frac{\omega\Delta t}{2}\right]$</p>	<p>Summed Wave: $s_{VSRT}(t) = 2\sin\left[\frac{2\omega t - \omega\Delta t - (\omega_a + \omega_b)t}{2}\right] \cos\left[\frac{\omega\Delta t + (\omega_b - \omega_a)t}{2}\right]$ Note that if $\omega_a = \omega_b = 0$, then $s_{VSRT}(t) = s(t)$.</p>
<p>Maximum <u>destructive</u> interference ($S_1(t) + S_2(t) = 0$) happens when:</p> <ul style="list-style-type: none"> • $\Delta t = \frac{D}{c} \sin\theta = \frac{(2n+1)\pi}{\omega}$ - • $x = D\sin\theta = (n + \frac{1}{2})\lambda$ - i.e. when the waves are out of phase by 180°, or an odd number of half wavelengths <p>Maximum <u>constructive</u> interference happens when:</p> <ul style="list-style-type: none"> • $\Delta t = \frac{D}{c} \sin\theta = \frac{2n\pi}{\omega}$ <p>$x = D\sin\theta = n\lambda$, i.e. when the waves are in phase, or spaced apart by an integer number of wavelengths</p>	<p>There is no time during which we have pure constructive or destructive interference.</p> <p>The signal that is read from the computer for one source averaged over time is,</p> $s_{VSRT}^2(t) = 1 + \cos[\omega\Delta t + (\omega_b - \omega_a)t]$

Interestingly, we can get constructive and destructive interference with the VSRT, but we have to use **two** CFLs.

CASE 3: VSRT WITH TWO CFL SOURCES

If we now use the VSRT with two CFLs, *source₁* and *source₂*, the registered signal is the sum of 4 sine waves:

$$\begin{aligned} s_{2 \text{ sources}}(t) &= \text{Source}_1(t) + \text{Source}_2(t) \\ &= \{ \sin[(\omega - \omega_a)t] + \sin[(\omega - \omega_b)t - \omega \Delta t_1] \} \\ &\quad + \{ \sin[(\omega - \omega_a)t] + \sin[(\omega - \omega_b)t - \omega \Delta t_2] \} \end{aligned}$$

where the notably different terms are $\Delta t_{1,2}$ representing time delays associated with different locations of source 1 or 2.

Using trig identity 1 listed in table 1, this expression simplifies to

$$\begin{aligned} s_{2 \text{ sources}}(t) &= 2 \sin \left[\frac{2\omega t - \omega \Delta t_1 - (\omega_a + \omega_b)t}{2} \right] \cos \left[\frac{\omega \Delta t_1 + (\omega_b - \omega_a)t}{2} \right] \\ &\quad + 2 \sin \left[\frac{2\omega t - \omega \Delta t_2 - (\omega_a + \omega_b)t}{2} \right] \cos \left[\frac{\omega \Delta t_2 + (\omega_b - \omega_a)t}{2} \right] \end{aligned}$$

Again, the only terms that are different from $s_{VSRT}(t)$ are the delay times $\Delta t_{1,2}$.

If we square the signal and time average the terms like we did in CASE 2, the expression for $s_{2 \text{ sources}}^2(t)$ reduces to

$$s_{2 \text{ sources}}^2(t) = 2 + \cos(\omega \Delta t_1 + (\omega_b - \omega_a)t) + \cos(\omega \Delta t_2 + (\omega_b - \omega_a)t) .$$

Using trig identity 4 from table 1, we simplify further so that

$$s_{2 \text{ sources}}^2 = 2 \cos \left[\frac{\omega(\Delta t_1 + \Delta t_2)}{2} + (\omega_b - \omega_a)t \right] \cos \left[\frac{\omega(\Delta t_1 - \Delta t_2)}{2} \right] + 2$$

Notice the similarities between $s_{2 \text{ sources}}^2$ and $s(t) = 2 \sin \left[\frac{2\omega t - \omega \Delta t}{2} \right] \cos \left[\frac{\omega \Delta t}{2} \right]$ from CASE 1. Most notably both have a term that does not depend on time, introducing an opportunity for constructive and destructive interference. These similarities are summarized in Table 4.

Table 4: Summary of how a VSRT with two CFLs can emulate a simple interferometer

<p>Simple Interferometer with a Single Source</p> <ul style="list-style-type: none"> ✓ single source infinitely far away at an angle θ (source position measured as angular deviation from normal) ✓ two collectors a distance D apart 	<p>VSRT with Two CFLs</p> <ul style="list-style-type: none"> ✓ Two sources infinitely far away at an angle $\theta_{1,2}$ arriving with time delays $\Delta t_{1,2}$ ✓ two collectors a distance D apart, ✓ Collectors adding signals with DIFFERENT frequencies, where one is offset by ω_a and the other by ω_b
<p><i>Signals are simply added</i></p>	<p><i>Signals are frequency shifted then added</i></p>
<p>Summed Wave: $s(t) = S_1(t) + S_2(t)$ $s(t) = 2\sin\left[\frac{2\omega t - \omega\Delta t}{2}\right] \cos\left[\frac{\omega\Delta t}{2}\right]$</p>	<p>Summed Wave: $s_{2\text{ sources}}^2 = 2\cos\left[\frac{\omega(\Delta t_1 + \Delta t_2)}{2} + (\omega_b - \omega_a)t\right] \cos\left[\frac{\omega(\Delta t_1 - \Delta t_2)}{2}\right] + 2$</p>
<p>Maximum <u>destructive</u> interference ($S_1(t) + S_2(t) = 0$) happens when:</p> <ul style="list-style-type: none"> • $\Delta t = \frac{D}{c} \sin\theta = \frac{(2n+1)\pi}{\omega}$ • $x = D\sin\theta = \left(n + \frac{1}{2}\right)\lambda$ - i.e. when the waves are out of phase by 180°, or an odd number of half wavelengths apart <p>Maximum <u>constructive</u> interference happens when:</p> <ul style="list-style-type: none"> • $\Delta t = \frac{D}{c} \sin\theta = \frac{2n\pi}{\omega}$ <p>$x = D\sin\theta = n\lambda$, i.e. when the light waves are in phase, or spaced apart by an integer number of wavelengths apart</p>	<p>Maximum <u>destructive</u> interference happens when:</p> <ul style="list-style-type: none"> • $\Delta t_1 - \Delta t_2 = \frac{D}{c} (\sin\theta_1 - \sin\theta_2) = \frac{(2n+1)\pi}{\omega}$ • $x_1 - x_2 = D(\sin\theta_1 - \sin\theta_2) = \left(n + \frac{1}{2}\right)\lambda$ - i.e. when the wavelengths are out of phase by 180°, or an odd number of half wavelengths apart <p>Maximum <u>constructive</u> interference happens when:</p> <ul style="list-style-type: none"> • $\Delta t_1 - \Delta t_2 = \frac{D}{c} (\sin\theta_1 - \sin\theta_2) = \frac{n2\pi}{\omega}$ • $x_1 - x_2 = D(\sin\theta_1 - \sin\theta_2) = n\lambda$, i.e. when the wavelengths are in phase, or an integer number of wavelengths apart

In summary: If we now look at two sources with the VSRT, we find that the VSRT acts like the simple interferometer discussed in CASE 1. The $s_{2\text{ sources}}^2(t)$ expression for CASE 3 is most similar to the $s(t)$ expression for CASE 1. The cosine term at the end of the expression for $s_{2\text{ sources}}^2(t)$ does not depend on t , so it is possible for us to have constructive and destructive interference. Instead of Δt , we now have the difference in the time delays between the two sources, $\Delta t_1 - \Delta t_2$, so once again, as in CASE 1, the interference depends on the location of the sources and the distance between the collectors. (See also VSRT memo 34 (Rogers, VSRT Memo #34, 2008) using an alternate analysis using the concept of the correlation of signals in radio interferometry. From this viewpoint it is shown how the square law detector provides the correlation of the signals from two LNBFs.)

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